

Coupling topological field theories to topological backgrounds

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Based on:



- [1411.6635](#) with C. Imbimbo (w.i.p. with C. Imbimbo, S.J. Rey and J. Bae)

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Cosmology 2015

Main result of the last few years




Exact results for SUSY QFTs in different dimensions and for different manifolds

Two main technical ingredients:

- Localization  Exact results [Pestun,...]
- Coupling to gravity  Manifolds with SUSY
[Festuccia-Seiberg,...]

An algorithmical recipe


The two methods combined give a clear **recipe**:

1. Flat space SUSY  relevant SUGRA
2. SUSY backgrounds for SUGRA
3. Rigid SUSY  SUGRA_{freezed} + matter
4. Localization  exact results

A couple of remarks

- Localization is often deeply related to **topological field theories**

Examples:


- 
- 1) Nekrasov instanton partition function
 - 2) Chern-Simons theory using SUSY
 - 3) 2d GLSM to compute GW invariants

- Localization relies on a **cohomological** formulation of SUSY

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Examples:

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- 1) Nekrasov instanton partition function
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- Localization relies on a **cohomological** formulation of SUSY

Can we find a formulation which is cohomological from the very beginning, i.e. from supergravity?

A different perspective

At least in 3d, a new approach can be found:

A different perspective

At least in 3d, a new approach can be found:

1. Consider CS + topological gravity



hidden, “**topological**”, SUSY appears

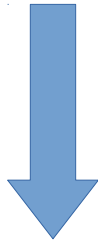
2. Topological **rigid** backgrounds are easily found
3. Exact computations are much easier!
4. Hints that the approach can be **generalized**

Plan of the talk

1. Introduction
2. CS and 3d VM coupled to backgrounds
3. Exact results

3d CS and vector multiplet

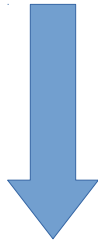
- First question: What gravity we need?



3d, topological gravity

3d CS and vector multiplet

- **First question:** What gravity we need?



3d, **topological** gravity

$$s g_{\mu\nu} = \psi_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu}$$

$$s \psi_{\mu\nu} = \mathcal{L}_\gamma g_{\mu\nu} - \mathcal{L}_\xi \psi_{\mu\nu}$$

$$s \xi^\mu = \gamma^\mu - \frac{1}{2} \mathcal{L}_\xi \xi^\mu$$

$$s \gamma^\mu = -\mathcal{L}_\xi \gamma^\mu$$

$g_{\mu\nu}$ metric

$\psi_{\mu\nu}$ twisted gravitino

ξ^μ vector, ghost-field

γ^μ vector, ghost-for-ghost field

- The BV-CS theory coupled to gravity found by [\[Imbimbo\]](#)



Let us review how it works!

$$s A = -Dc - \mathcal{L}_\xi A + \dots$$

Coupling to
diffeomorphisms and
require nilpotency

- The BV-CS theory coupled to gravity found by [Imbimbo]



Let us review how it works!

$$s A = -Dc - \mathcal{L}_\xi A + \dots$$

Coupling to
diffeomorphisms and
require nilpotency

Invariant action:

$$\Gamma_{CS+t.g.} = \Gamma_{CS} + \frac{1}{2} \int_{M_3} i_\gamma(\tilde{A}) \tilde{A}$$

symmetries



- **Symmetry** transformations:

$$\begin{aligned}
 s\,c &= -c^2 - \mathcal{L}_\xi c + i_\gamma(A) & s\,\tilde{A} &= -[\tilde{A}, c] - \mathcal{L}_\xi \tilde{A} - F + i_\gamma(\tilde{c}) \\
 s\,A &= -D\,c - \mathcal{L}_\xi A + i_\gamma(\tilde{A}) & s\,\tilde{c} &= -[\tilde{c}, c] - \mathcal{L}_\xi \tilde{c} - D\,\tilde{A}
 \end{aligned}$$

That can be understood better!

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- **Rigid** case:

$$\delta_0 \equiv S_0 + d$$

$$\mathcal{A} \equiv c + A + \tilde{A} + \tilde{c}$$



$$\delta_0 \mathcal{A} + \mathcal{A}^2 = 0$$

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- **Coupled** case:

$$\delta \equiv s + d + \mathcal{L}_\xi - i_\gamma$$

$$\mathcal{A} \equiv c + A + \tilde{A} + \tilde{c}$$



$$\delta \mathcal{A} + \mathcal{A}^2 = 0$$

Just a **deformation** of the **coboundary** operator!

Comparison with physical models

The **twisted** version of physical SUSY vector multiplet, have been computed in a **fixed** background.

[Kallen]



We can compare with our result!

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When specialized to **fixed**, **supersymmetric**, backgrounds our results are equivalent to the physical ones.

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When specialized to **fixed**, **supersymmetric**, backgrounds our results are equivalent to the physical ones.

We realized **coupling** of vector multiplet to backgrounds. Our coupling is valid also **away** from SUSY backgrounds

Exact results

- We pass to discuss many **exact** results at disposal:

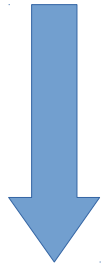
1. Supersymmetric backgrounds

2. Moduli characterization

3. Moduli dependence through topological anomaly

Supersymmetric backgrounds

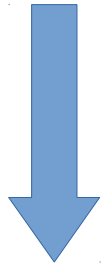
- Standard recipe: SUSY backgrounds from gravity



$$\begin{aligned}\text{fermions} &= 0 \\ \delta \text{ fermions} &= 0\end{aligned}$$

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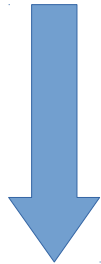
Come back to topological gravity

$$\begin{aligned}\psi_{\mu\nu} &= 0 \\ s \psi_{\mu\nu} = \mathcal{L}_\gamma g_{\mu\nu} &= 0\end{aligned}$$

→ γ has to be Killing!

Supersymmetric backgrounds

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→ γ has to be **Killing**!

In this formalism SUSY backgrounds are straightforwardly identified with **Seifert manifolds**

Moduli characterization

- SUSY backgrounds: pairs $(\bar{g}_{\mu\nu}, \bar{\gamma})$ such that

$$\mathcal{L}_{\bar{\gamma}} \bar{g}_{\mu\nu} \equiv \bar{D}_{\mu} \bar{\gamma}_{\nu} + \bar{D}_{\nu} \bar{\gamma}_{\mu} = 0$$

Let us explore SUSY deformations $\{\bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \gamma^{\mu} + \delta \gamma^{\mu}\}$

$$\mathcal{L}_{\bar{\gamma}} \delta g_{\mu\nu} + \mathcal{L}_{\delta \gamma} \bar{g}_{\mu\nu} = 0$$

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Moduli space:

$$\ker \varphi = \{\gamma : [\gamma, \bar{\gamma}] = 0, \mathcal{L}_{\gamma} \bar{g}_{\mu\nu} = 0\}$$

$$\mathcal{M} \simeq \ker \varphi / \sim$$

$$\gamma' \sim \gamma + \alpha \bar{\gamma}$$

Moduli dependence through an anomaly

- Anomalous topological **Ward identity**: Anomaly 3-form

The diagram consists of a blue rectangular box containing the equation $S \log Z[g_{\mu\nu}, \psi_{\mu\nu}, \gamma^\mu] = \int_{M_3} A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}]$. A dashed arrow originates from the text 'topological gravity BRST operator' below the box and points to the $S \log Z$ term inside the box. Another dashed arrow originates from the text 'Anomaly 3-form' above the box and points to the $A_1^{(3)}$ term inside the box.

$$S \log Z[g_{\mu\nu}, \psi_{\mu\nu}, \gamma^\mu] = \int_{M_3} A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}]$$

topological gravity
BRST operator

$$A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}] = \frac{c}{6} \epsilon^{\mu\nu\rho} R_\mu^\alpha D_\nu \psi_{\rho\alpha} d^3x$$

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Since this is an anomaly
we have not possible
counterterms to cancel it

$$S \int_{M_3} \Gamma[g_{\mu\nu}] = \int_{M_3} A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}]$$

Not a 3-form under
3d **diffeomorphisms**

- SUSY backgrounds are **Seifert manifolds**:

$$g_{\mu\nu} \rightarrow (g_{ij}, a_i, \sigma)$$

$$(ds)_M^2 = e^\sigma k(a_i) \otimes k(a_i) + g_{ij} dx^i \otimes dx^j$$

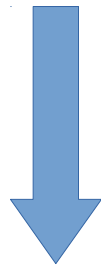
3d metric parametrized by
a **2d metric**, a U(1) **vector**
and a **scalar**

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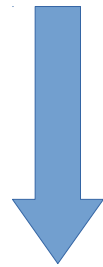
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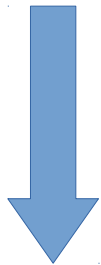
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In this **renormalization scheme**
anomaly can be removed!



2d diffeomorphisms
+
U(1) gauge invariance

Let's see how!!



- In Seifert scheme anomaly is **removable**

$$\mathcal{A} = s_{Seif} \Gamma_{WZ}^{Seif} [g_{ij}, \sigma, a_i]$$

Seifert topological BRST

Wess-Zumino action

$$\Gamma_{WZ}^{Seif} [g_{ij}, \sigma, a_i] = \frac{1}{2} \sqrt{g} e^{2\sigma} f^3 - \sqrt{g} \frac{1}{2} e^\sigma f \hat{R} - \frac{1}{2} \sqrt{g} e^\sigma f D^2 \sigma$$

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Seifert effective action
not anomalous


$$\tilde{\Gamma}^{Seif} \equiv \Gamma^{Seif} - \Gamma_{WZ}^{Seif}$$

$$s_{Seif} \tilde{\Gamma}^{Seif} = 0$$

An application: squashed spheres

- Consider the squashed three-sphere

$$ds^2 = (\sin^2 \theta + b^4 \cos^2 \theta) d\theta^2 + \cos^2 \theta d\phi_1^2 + b^4 \sin^2 \theta d\phi_2^2$$

squashing parameter.
Seifert modulus

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Moduli dependence:

All the dependence comes from the WZ-term

$$\Gamma_{WZ}^{Seif} = -\frac{c}{6} (2\pi)^2 \left(b^2 + \frac{1}{b^2}\right) \text{ matches with } \text{localization} \text{ result!!}$$

Conclusions

1. A different approach to introduce SUSY
2. Topological SUSY instead of spinors
 Much **simpler** formalism
3. Results that match with spinorial approach

Outlook

(w.i.p. with C. Imbimbo, S. J. Rey and J. B. Bae)

1. Can we insert **matter**? More general topological backgrounds could be necessary
2. Generalization to other dimensions
(w.i.p. in 2d theories)
3. Can we always apply this strategy?
4.