Coupling topological field theories to topological backgrounds

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Based on:

- 1411.6635 with C. Imbimbo (w.i.p. with C. Imbimbo, S.J. Rey and J. Bae)

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Main result of the last few years

Exact results for SUSY QFTs in different dimensions and for different manifolds

Two main technical ingredients:

- Localization → Exact results [Pestun,...]
- Coupling to gravity → Manifolds with SUSY

[Festuccia-Seiberg,...]

An algorithmical recipe

The two methods combined give a clear recipe:

1. Flat space SUSY → relevant SUGRA

2. SUSY backgrounds for SUGRA

3. Rigid SUSY → $\text{SUGRA}_{\text{freezed}}$ + matter

4. Localization → exact results

A couple of remarks

- Localization is often deeply related to topological field theories

Examples:

1) Nekrasov instanton partition function
2) Chern-Simons theory using SUSY
3) 2d GLSM to compute GW invariants

- Localization relies on a cohomological formulation of SUSY

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Examples:

1) Nekrasov instanton partition function
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- Localization relies on a cohomological formulation of SUSY

Can we find a formulation which is cohomological from the very beginning, i.e. from supergravity?

A different perspective

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At least in 3d, a new approach can be found:

1. Consider **CS + topological gravity**

   → hidden, “topological”, SUSY appears

2. Topological rigid backgrounds are easily found

3. Exact computations are much easier!

4. Hints that the approach can be generalized

Plan of the talk

1. Introduction

2. CS and 3d VM coupled to backgrounds

3. Exact results

3d CS and vector multiplet

- First question: What gravity we need?

↓

3d, topological gravity

3d CS and vector multiplet

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3d, topological gravity

$$
\begin{aligned}
s\, g_{\mu\nu} &= \psi_{\mu\nu} - \mathcal{L}_\xi\, g_{\mu\nu} \\
s\, \psi_{\mu\nu} &= \mathcal{L}_\gamma\, g_{\mu\nu} - \mathcal{L}_\xi\, \psi_{\mu\nu} \\
s\, \xi^\mu &= \gamma^\mu - \frac{1}{2}\mathcal{L}_\xi\, \xi^\mu \\
s\, \gamma^\mu &= -\mathcal{L}_\xi\, \gamma^\mu
\end{aligned}
$$

$g_{\mu\nu}$ metric

$\psi_{\mu\nu}$ twisted gravitino

ξ^μ vector, ghost-field

γ^μ vector, ghost-for-ghost field

- The BV-CS theory coupled to gravity found by [Imbimbo]

Let us review how it works!

$$s\,A = -Dc - \mathcal{L}_\xi A + \ldots$$

Coupling to diffeomorphisms and require nilpotency

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Coupling to diffeomorphisms and require nilpotency

Invariant action:

$$\Gamma_{CS+t.g.} = \Gamma_{CS} + \frac{1}{2}\int_{M_3} i_\gamma(\tilde{A})\,\tilde{A}$$

symmetries

- Symmetry transformations:

$$s\,c = -c^2 - \mathcal{L}_\xi\, c + i_\gamma(A) \qquad s\,\tilde{A} = -[\tilde{A}, c] - \mathcal{L}_\xi\, \tilde{A} - F + i_\gamma(\tilde{c})$$

$$s\,A = -D\,c - \mathcal{L}_\xi\, A + i_\gamma(\tilde{A}) \qquad s\,\tilde{c} = -[\tilde{c}, c] - \mathcal{L}_\xi\, \tilde{c} - D\,\tilde{A}$$

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- Rigid case:

$$\delta_0 \equiv S_0 + d$$

$$\mathcal{A} \equiv c + A + \tilde{A} + \tilde{c}$$

$$\Rightarrow \quad \boxed{\delta_0 \mathcal{A} + \mathcal{A}^2 = 0}$$

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- Coupled case:

$$\delta \equiv s + d + \mathcal{L}_\xi - i_\gamma$$

$$\mathcal{A} \equiv c + A + \tilde{A} + \tilde{c}$$

$\Rightarrow$ $\boxed{\delta\mathcal{A} + \mathcal{A}^2 = 0}$

Just a deformation of the coboundary operator!

Comparison with physical models

The twisted version of physical SUSY vector multiplet, have been computed in a fixed background.
[Kallen]

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When specialized to fixed, supersymmetric, backgrounds our results are equivalent to the physical ones.

We realized coupling of vector multiplet to backgrounds.
Our coupling is valid also away from SUSY backgrounds

Exact results

- We pass to discuss many exact results at disposal:

1. Supersymmetric backgrounds

2. Moduli characterization

3. Moduli dependence through topological anomaly

Supersymmetric backgrounds

- Standard recipe: SUSY backgrounds from gravity

$$\begin{aligned} \text{fermions} &= 0 \\ \delta\, \text{fermions} &= 0 \end{aligned}$$

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Come back to topological gravity

$$\psi_{\mu\nu} = 0$$
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→ γ has to be Killing!

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$\longrightarrow$ γ has to be Killing!

In this formalism SUSY backgrounds are straightforwardly identified with Seifert manifolds

Moduli characterization

- SUSY backgrounds: pairs $(\bar{g}_{\mu\nu},\, \bar{\gamma})$ such that

$$\mathcal{L}_{\bar{\gamma}}\, \bar{g}_{\mu\nu} \equiv \bar{D}_\mu\, \bar{\gamma}_\nu + \bar{D}_\nu\, \bar{\gamma}_\mu = 0$$

Let us explore SUSY deformations $\{\bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \gamma^\mu + \delta\gamma^\mu\}$

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Moduli space:

$$\ker\varphi = \{\gamma : [\gamma, \bar{\gamma}] = 0,\ \mathcal{L}_\gamma \bar{g}_{\mu\nu} = 0\}$$

$$\mathcal{M} \simeq \ker\varphi / \sim$$

$$\gamma' \sim \gamma + \alpha\bar{\gamma}$$

Moduli dependence through an anomaly

- Anomalous topological Ward identity:

Anomaly 3-form

$$S \log Z[g_{\mu\nu}, \psi_{\mu\nu}, \gamma^{\mu}] = \int_{M_3} A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}]$$

topological gravity
BRST operator

$$A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}] = \frac{c}{6}\, \epsilon^{\mu\nu\rho}\, R_{\mu}^{\alpha}\, D_{\nu}\, \psi_{\rho\alpha}\, d^3x$$

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Since this is an anomaly
we have not possible
counterterms to cancel it

$$S \int_{M_3} \Gamma[g_{\mu\nu}] = \int_{M_3} A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}]$$

Not a 3-form under
3d diffeomorphisms

- SUSY backgrounds are Seifert manifolds:

$$g_{\mu\nu} \to (g_{ij}\,,\, a_i\,, \sigma\,)$$

$$(ds)^2_M = \mathrm{e}^{\sigma}\, k(a_i) \otimes k(a_i) + g_{ij}\, dx^i \otimes\, dx^j$$

3d metric parametrized by a 2d metric, a U(1) vector and a scalar

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2d diffeomorphisms
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In this renormalization scheme anomaly can be removed!

Let's see how!!

- In Seifert scheme anomaly is removable

$$\mathcal{A} = s_{Seif}\, \Gamma^{Seif}_{WZ}[g_{ij}, \sigma, a_i]$$

Seifert topological BRST

Wess-Zumino action

$$\Gamma^{Seif}_{WZ}[g_{ij}, \sigma, a_i] = \frac{1}{2}\sqrt{g}\,\mathrm{e}^{2\,\sigma}\, f^3 - \sqrt{g}\,\frac{1}{2}\,\mathrm{e}^{\sigma}\, f\,\hat{R} - \frac{1}{2}\sqrt{g}\,\mathrm{e}^{\sigma}\, f\, D^2\,\sigma$$

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Seifert effective action

not anomalous

$$\tilde{\Gamma}^{Seif} \equiv \Gamma^{Seif} - \Gamma^{Seif}_{WZ}$$

$$s_{Seif}\, \tilde{\Gamma}^{Seif} = 0$$

An application: squashed spheres

- Consider the squashed three-sphere

$$ds^2 = (\sin^2\theta + b^4\ \cos^2\theta)\, d\theta^2 + \cos^2\theta\, d\phi_1^2 + b^4\ \sin^2\theta\, d\phi_2^2$$

squashing parameter.
Seifert modulus

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The anomaly vanishes $\Rightarrow$ $A_1^{(3)}(b) = 0$

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squashing parameter.
Seifert modulus

The anomaly vanishes → $A_1^{(3)}(b) = 0$

Moduli dependence:
All the dependence comes from the WZ-term

$$\Gamma_{WZ}^{Seif} = -\frac{c}{6}(2\,\pi)^2\,(b^2 + \frac{1}{b^2})$$

matches with localization result!!

Conclusions

1. A different approach to introduce SUSY

2. Topological SUSY instead of spinors

→ Much simpler formalism

3. Results that match with spinorial approach

Outlook (w.i.p. with C. Imbimbo, S. J. Rey and J. B. Bae)

1. Can we insert matter? More general topological backgrounds could be necessary

2. Generalization to other dimensions
   (w.i.p. in 2d theories)

3. Can we always apply this strategy?

4. ………..